

# HOMOLOGY OF EQUIVARIANT VECTOR FIELDS

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**ABSTRACT.** Let  $K$  be a compact Lie group. We compute the abelianization of the Lie algebra of equivariant vector fields on a smooth  $K$ -manifold  $X$ . We also compute the abelianization of the Lie algebra of strata preserving smooth vector fields on the quotient  $X/K$ .

## 1. INTRODUCTION

1.1. K. Abe and K. Fukui [AbFu2] have considered the first homology group (abelianization) of the group of equivariant smooth diffeomorphisms of a smooth  $K$ -manifold  $X$ , where  $K$  is finite. They also computed the abelianization for the diffeomorphisms of the quotient orbifold  $X/K$ . Our results below are the analogues of their results for vector fields in the case that  $K$  is a compact Lie group. The vector fields are, in a sense, the Lie algebras of the relevant diffeomorphism groups, so, hopefully, our results indicate that one should be able to generalize the Abe-Fukui results. There are already generalizations in some cases [AbFu1].

1.2. Let  $X$  be a smooth  $K$ -manifold where  $K$  is compact. Let  $\mathcal{X}^\infty(X)$  denote the Lie algebra of smooth vector fields on  $X$  and let  $\mathcal{X}_c^\infty(X)$  denote the subalgebra of vector fields with compact support. If  $X$  is algebraic, then  $\mathcal{X}(X)$  will denote the polynomial vector fields on  $X$ . By  $\mathcal{X}^\infty(X)^K$ , etc. we mean the  $K$ -invariant elements in  $\mathcal{X}^\infty(X)$ , etc. We will state most of our results for  $\mathcal{X}_c^\infty(X)^K$ ; the corresponding results for  $\mathcal{X}^\infty(X)^K$  follow easily from our techniques.

If  $\mathfrak{g}$  is a Lie algebra, we denote by  $\mathcal{H}(\mathfrak{g})$  the abelianization  $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ . We denote the Lie algebras of compact Lie groups  $K$ ,  $H$ , etc. by the corresponding gothic letters  $\mathfrak{k}$ ,  $\mathfrak{h}$ , etc.

1.3. Let  $x \in X$ . Then we have the isotropy group  $K_x$  and its slice representation on  $W_x := T_x X / T_x(Kx)$  where  $Kx$  denotes the  $K$ -orbit through  $x$ . We say that the orbit  $Kx$  is *isolated* if  $W_x^{K_x} = (0)$ . It follows from the differentiable slice theorem that  $Kx$  is isolated if and only if all isotropy groups  $K_y$  of points  $y$  near  $x$ ,  $Ky \neq Kx$ , are conjugate to a proper subgroup of  $K_x$ . There is then a discrete subset  $\{x_i\}_{i \in I}$  of  $X$  (possibly empty) where we choose one point from each isolated orbit. Let  $H_i$  denote  $K_{x_i}$  and set  $W_i := W_{x_i}$ ,  $i \in I$ .

**Theorem 1.4.** *Let  $X$  and the  $x_i$ ,  $H_i$  and  $W_i$  be as above. Then*

$$\mathcal{H}(\mathcal{X}_c^\infty(X)^K) \simeq \bigoplus_i \mathcal{H}(\mathfrak{k}^{H_i} / \mathfrak{h}_i^{H_i}) \bigoplus_i \mathcal{H}(\text{End}(W_i)^{H_i}).$$

**Theorem 1.5.** *Let  $H$  be a compact Lie group and  $V$  an  $H$ -module where  $V^H = (0)$ . Write  $V = \bigoplus_{j=1}^m n_j V_j$  where the  $V_j$  are irreducible and pairwise non-isomorphic and  $n_j V_j$  denotes the direct sum of  $n_j$  copies of  $V_j$ . Let  $l$  denote the number of  $V_j$  such that  $\text{End}(V_j)^H \simeq \mathbb{C}$  and let  $Z(\text{End}(V)^H)$  denote the center of  $\text{End}(V)^H$ . Then*

$$\mathcal{H}(\text{End}(V)^H) \simeq Z(\text{End}(V)^H) = \bigoplus_j Z(\text{End}(n_j V_j)^H) \simeq \mathbb{R}^{m-l} \oplus \mathbb{C}^l.$$

2000 *Mathematics Subject Classification.* 22C05, 57R25, 57S15.

Partially supported by NSA grant H98230-04-01-0070.

Let  $\mathcal{X}_c^\infty(X/K)$  denote the Lie algebra of compactly supported smooth strata preserving vector fields on  $X/K$  (see §4 for definitions).

**Theorem 1.6.** *Let  $X$  and the  $x_i$ ,  $H_i$  and  $W_i$  be as above. Then*

$$\mathcal{H}(\mathcal{X}_c^\infty(X/K)) \simeq \bigoplus_i (Z(\text{End}(W_i)^{H_i})/\mathfrak{s}_i)$$

where each  $\mathfrak{s}_i$  is the Lie algebra of a torus  $S_i$  lying in  $Z(\text{End}(W_i)^{H_i})$ .

We will say more about the  $S_i$  in §4.

1.7. This work was done while attending the conference “Diffeomorphisms and Related Fields” held at Shinshu University, December 2005. The author thanks professors K. Abe and K. Fukui for the invitation and for their wonderful hospitality.

## 2. VANISHING OF ABELIANIZATIONS

2.1. In the following, let  $\mathcal{B}_c^\infty(X)^K$  denote  $[\mathcal{X}_c^\infty(X)^K, \mathcal{X}_c^\infty(X)^K]$  and let  $\mathcal{C}_c^\infty(X)^K$  denote the compactly supported smooth functions on  $X$ . Our first goal is to show that  $\mathcal{H}(\mathcal{X}_c^\infty(X \times \mathbb{R})^K)$  is zero.

**Lemma 2.2.** *Let  $A \in \mathcal{X}_c^\infty(X)^K$  and  $B \in \mathcal{X}^\infty(X)^K$ . Then  $[A, B] \in \mathcal{B}_c^\infty(X)^K$ .*

*Proof.* Let  $g \in \mathcal{C}_c^\infty(X)^K$  be identically 1 on a neighborhood of  $\text{supp } A$ . Then  $[A, gB] = g[A, B] + A(g)B = [A, B] \in \mathcal{B}_c^\infty(X)^K$ .  $\square$

**Proposition 2.3.** *Let  $K$  act on  $X \times \mathbb{R}$  with the given action on  $X$  and the trivial action on  $\mathbb{R}$ . Then  $\mathcal{H}(\mathcal{X}_c^\infty(X \times \mathbb{R})^K) = 0$ .*

*Proof.* Let  $t$  denote the usual coordinate function on  $\mathbb{R}$  and let  $g \in \mathcal{C}_c^\infty(X \times \mathbb{R})^K$ . We show that  $g \frac{d}{dt} \in \mathcal{B}_c^\infty(X \times \mathbb{R})^K$ . For  $x \in X$  and  $s \in \mathbb{R}$  set  $h(x, s) = \int_0^s g(x, u) du$ . Then  $h$  is smooth and  $K$ -invariant. Let  $f \in \mathcal{C}_c^\infty(X \times \mathbb{R})^K$ . Then

$$\begin{aligned} [f \frac{d}{dt}, h \frac{d}{dt}] &= f \frac{dh}{dt} \frac{d}{dt} - h \frac{df}{dt} \frac{d}{dt} \text{ and} \\ [\frac{d}{dt}, fh \frac{d}{dt}] &= f \frac{dh}{dt} \frac{d}{dt} + h \frac{df}{dt} \frac{d}{dt}. \end{aligned}$$

Hence  $2fg \frac{d}{dt} \in \mathcal{B}_c^\infty(X \times \mathbb{R})^K$ . If  $f$  equals 1/2 on a neighborhood of  $\text{supp } g$ , we obtain that  $g \frac{d}{dt} \in \mathcal{B}_c^\infty(X \times \mathbb{R})^K$ .

Now suppose that  $A \in \mathcal{X}_c^\infty(X \times \mathbb{R})^K$ . By our result above, we can assume that  $A$  annihilates  $t$ . Set  $B(x, s) = \int_0^s A(x, u) du$  and let  $g \in \mathcal{C}_c^\infty(X \times \mathbb{R})^K$  equal 1 on a neighborhood of  $\text{supp } A$ . Then  $[g \frac{d}{dt}, B] = gA - B(g) \frac{d}{dt}$ . We already know that  $B(g) \frac{d}{dt} \in \mathcal{B}_c^\infty(X \times \mathbb{R})^K$ , hence  $A \in \mathcal{B}_c^\infty(X \times \mathbb{R})^K$ . Thus  $\mathcal{H}(\mathcal{X}_c^\infty(X \times \mathbb{R})^K) = 0$ .  $\square$

2.4. Let  $H$  be a closed subgroup of  $K$  and  $W$  an  $H$ -module. Then we have the twisted product  $K *^H W$  which is the quotient  $(K \times W)/H$  where  $h(k, w) = (kh^{-1}, hw)$ ,  $h \in H$ ,  $k \in K$  and  $w \in W$ . We denote the image of  $(k, w) \in K \times W$  in  $K *^H W$  by  $[k, w]$ . Note that  $K *^H W$  is naturally a  $K$ -vector bundle and a real algebraic  $K$ -variety [Schw3].

Let  $H \rightarrow \text{GL}(W)$  be the slice representation at a point  $x \in X$ . By the differentiable slice theorem, a  $K$ -neighborhood of  $Kx$  in  $X$  is  $K$ -diffeomorphic to  $K *^H W$ . By Proposition 2.3,  $\mathcal{H}(\mathcal{X}_c^\infty(K *^H W)^K) = 0$  if  $W^H \neq (0)$ .

Let  $F$  be a closed  $K$ -stable subset of  $X$ . We say that  $\mathcal{H}(\mathcal{X}_c^\infty(X)^K)$  is supported on  $F$  if  $\mathcal{H}(\mathcal{X}_c^\infty(X \setminus F)^K) = 0$ . Using a partition of unity argument we can show

**Corollary 2.5.** *Let  $F = \{x \in X \mid W_x^{K_x} = 0\}$ . Then  $\mathcal{H}(\mathcal{X}_c^\infty(X)^K)$  is supported on  $F$ .*

### 3. LOCAL COMPUTATIONS

3.1. Our results above show that there is a discrete set of orbits  $\{Kx_i\}$  such that

$$\mathcal{H}(\mathcal{X}_c^\infty(X)^K) \simeq \bigoplus_i \mathcal{H}(\mathcal{X}_c^\infty(K *^{H_i} W_i)^K)$$

where  $H_i = K_{x_i}$  and  $W_i$  is the slice representation of  $H_i$  at  $x_i$ . Thus it suffices to compute  $\mathcal{H}(\mathcal{X}_c^\infty(K *^H V)^K)$  where  $H$  is a closed subgroup of  $K$ ,  $V$  is an  $H$ -module and  $V^H = (0)$ . This computation is the content of the following theorem.

**Theorem 3.2.** *Let  $H$  and  $V$  be as above. Then*

$$\mathcal{H}(\mathcal{X}_c^\infty(K *^H V)) \simeq \mathcal{H}(\mathfrak{k}^H / \mathfrak{h}^H) \oplus \mathcal{H}(\text{End}(V)^H).$$

3.3. Our proof of the theorem requires several lemmas. Set  $Y := K *^H V$ . Then

$$\mathcal{X}(Y)^K \simeq \mathcal{X}(K \times V)^{K \times H} / (\mathcal{O}(K \times V) \mathfrak{h})^{K \times H}$$

(see [Schw2, §4]) where  $H$  has the diagonal action (see 2.4) on  $K \times V$  (inducing an action of  $\mathfrak{h}$ ) and  $\mathcal{O}(K \times V)$  denotes the polynomial functions on  $K \times V$ . Now

$$\mathcal{X}(K \times V)^{K \times H} \simeq (\mathcal{X}(K) \otimes \mathcal{O}(V) \oplus \mathcal{O}(K) \otimes \mathcal{X}(V))^{K \times H} \simeq (\mathfrak{k} \otimes \mathcal{O}(V))^H \oplus (1 \otimes \mathcal{X}(V))^H$$

while

$$(\mathcal{O}(K \times V) \mathfrak{h})^{K \times H} \simeq (\mathfrak{h} \otimes \mathcal{O}(V))^H.$$

3.4. We have the Euler operator  $E \in \mathcal{X}(V)^H$ , where if  $x_1, x_2, \dots$  are coordinate functions on  $V$ , then  $E = \sum_i x_i \frac{\partial}{\partial x_i}$ . By the isomorphisms above,  $E$  can be considered as a  $(K \times H)$ -invariant vector field on  $K \times V$  and as a  $K$ -invariant vector field on  $Y$ .

**Lemma 3.5.** *Let  $f \in \mathcal{C}^\infty(Y)^K$ . Then  $f = E(h)$  for some  $h \in \mathcal{C}^\infty(Y)^K$  if and only if  $f([e, 0]) = 0$ .*

*Proof.* Clearly the condition on  $f$  is necessary. Suppose that  $f([e, 0]) = 0$ . Since  $f$  is  $K$ -invariant, it is determined by its restriction  $g$  to  $\{[e, v] \mid v \in V\} \simeq V$ , where  $g$  is  $H$ -invariant. Set  $h(v) = \int_0^1 (1/t) g(tv) dt$ . Then  $h \in \mathcal{C}^\infty(V)^H$  since  $g(0) = 0$ . We have

$$E(h)(v) = \int_0^1 \frac{1}{t} \sum_i x_i \frac{\partial g}{\partial x_i}(tv) t dt = \int_0^1 \sum_i x_i \frac{\partial g}{\partial x_i}(tv) dt = \int_0^1 \frac{d}{dt} g(tv) dt = g(v) - g(0) = g(v).$$

□

**Corollary 3.6.** *Let  $g \in \mathcal{C}_c^\infty(Y)^K$  such that  $g([e, 0]) = 0$ . Then  $gE \in \mathcal{B}_c^\infty(Y)^K$ .*

*Proof.* By Lemma 3.5,  $g = E(h)$  for some  $h \in \mathcal{C}^\infty(Y)^K$ . Let  $f \in \mathcal{C}_c^\infty(Y)^K$  such that  $f$  is 1/2 in a neighborhood of  $\text{supp } g$ . Then, as in Proposition 2.3,

$$[E, fhE] + [fE, hE] = 2fE(h)E = 2fgE,$$

so that  $gE \in \mathcal{B}_c^\infty(Y)^K$ .

□

3.7. Since  $Y$  is real algebraic, the results in [Schw1, §6] show that  $\mathcal{X}^\infty(Y) \simeq \mathcal{C}^\infty(Y) \otimes_{\mathcal{O}(Y)} \mathcal{X}(Y)$ . For compactly supported sections we clearly have that  $\mathcal{X}_c^\infty(Y) = \mathcal{C}_c^\infty(Y) \mathcal{X}(Y)$ .

3.8. We have an  $E$ -eigenspace decomposition

$$\mathcal{X}(K \times V)^{K \times H} \simeq \bigoplus_{m \geq 0} (\mathfrak{k} \otimes \mathcal{O}(V)_m)^H \oplus (1 \otimes \mathcal{X}(V)_m^H)$$

and similarly for  $(\mathfrak{h} \otimes \mathcal{O}(V))^H$ . The weights that occur in  $\mathcal{X}(V)^H$  are all positive since  $V^H = (0)$ . We have an induced decomposition

$$\mathcal{X}(Y)^K = \bigoplus_{m \geq 0} \mathcal{X}(Y)_m^K.$$

*Remark 3.9.* Since the sum only contains terms for  $m \geq 0$ , an element of  $\mathcal{X}(Y)^K$  applied to an element of  $\mathcal{C}^\infty(Y)^K \simeq \mathcal{C}^\infty(V)^H$  always vanishes at  $[e, 0]$ .

**Lemma 3.10.** *Let  $A \in \mathcal{X}(Y)_m^K$  and let  $f \in \mathcal{C}_c^\infty(Y)^K$ . Then  $fA \in \mathcal{B}_c^\infty(Y)^K$  if*

- (1)  $m > 0$  or
- (2)  $f([e, 0]) = 0$ .

*Proof.* Suppose that  $m > 0$ . Then  $[(1/m)fE, A] = fA - (1/m)A(f)E$  where  $A(f)E \in \mathcal{B}_c^\infty(Y)^K$  by Corollary 3.6. Hence  $fA \in \mathcal{B}_c^\infty(Y)^K$ . If  $m = 0$  and  $f([e, 0]) = 0$ , then let  $h \in \mathcal{C}^\infty(Y)^K$  be such that  $E(h) = f$ , and let  $g \in \mathcal{C}_c^\infty(Y)^K$ . Then

$$[gE, hA] = gE(h)A - hA(g)E = gfA - hA(g)E,$$

where  $hA(g)E \in \mathcal{B}_c^\infty(Y)^K$  by Corollary 3.6. We may arrange that  $gfA = fA$ , so  $fA \in \mathcal{B}_c^\infty(Y)^K$ .  $\square$

*Proof of Theorem 3.2.* We first define a map of Lie algebras  $\varphi: \mathcal{X}_c^\infty(Y)^K \rightarrow \mathcal{X}(Y)_0^K$ . Let  $B = \sum_{i=1}^m f_i B_i \in \mathcal{X}_c^\infty(Y)^K$  where  $f_i \in \mathcal{C}_c^\infty(Y)^K$  and  $B_i \in \mathcal{X}(Y)_{m_i}^K$ ,  $i = 1, \dots, m$ . Define  $\varphi(B) := \sum_{m_i=0} f_i([e, 0])B_i \in \mathcal{X}(Y)_0^K$ . It is obvious that  $\varphi$  is surjective. Suppose that  $C, D \in \mathcal{X}(Y)^K$  are eigenvectors for  $E$  and that  $f, g \in \mathcal{C}_c^\infty(Y)^K$ . Then  $[fC, gD] = fC(g)D - gD(f)C + fg[C, D]$  where  $C(g)$  and  $D(f)$  vanish at  $[e, 0]$ . Thus  $\varphi([fC, gD]) = (fg)(0)\varphi([C, D]) = (fg)(0)[\varphi(C), \varphi(D)] = [\varphi(fC), \varphi(gD)]$ . Now  $\varphi$  induces  $\tilde{\varphi}: \mathcal{H}(\mathcal{X}_c^\infty(Y)^K) \rightarrow \mathcal{H}(\mathcal{X}(Y)_0^K)$ , which is again surjective. Suppose that  $B = \sum_i f_i B_i \in \text{Ker}(\tilde{\varphi})$  where the  $B_i$  are in  $\mathcal{X}(Y)_0^K$ . Then  $\varphi(B) = \sum_j [C_j, D_j]$  where  $C_j, D_j \in \mathcal{X}(Y)_0^K$  for all  $j$ . Let  $f \in \mathcal{C}_c^\infty(Y)^K$  such that  $f$  is 1 on a neighborhood of  $[e, 0]$ . Then  $B - \sum_j [fC_j, fD_j] \in \mathcal{B}_c^\infty(Y)^K$ . Hence  $\tilde{\varphi}$  is an isomorphism. From our equations in 3.3 it follows that  $\mathcal{H}(\mathcal{X}(Y)_0^K) \simeq \mathcal{H}(\mathfrak{k}^H/\mathfrak{h}^H) \oplus \mathcal{H}(\text{End}(V)^H)$ .  $\square$

*Proof of Theorem 1.4.* The theorem is immediate from 3.1 and Theorem 3.2  $\square$

*Proof of Theorem 1.5.* Let  $V = \bigoplus_{j=1}^m n_j V_j$  and  $H$  be as in 1.5. Then  $\text{End}(V)^H \simeq \bigoplus_j \text{End}(n_j V_j)^H$ . There are three cases to consider.

Case 1:  $\text{End}(V_j)^H \simeq \mathbb{R}$ . Then  $\text{End}(n_j V_j)^H \simeq \mathfrak{gl}(n_j, \mathbb{R})$  and  $\mathcal{H}(\mathfrak{gl}(n_j, \mathbb{R})) \simeq Z(\mathfrak{gl}(n_j, \mathbb{R})) \simeq \mathbb{R}$ .

Case 2:  $\text{End}(V_j)^H \simeq \mathbb{C}$ . Then  $\text{End}(n_j V_j)^H \simeq \mathfrak{gl}(n_j, \mathbb{C})$  and  $\mathcal{H}(\mathfrak{gl}(n_j, \mathbb{C})) \simeq Z(\mathfrak{gl}(n_j, \mathbb{C})) \simeq \mathbb{C}$ .

Case 3:  $\text{End}(V_j)^H \simeq \mathbb{H}$ , the quaternions. Then  $\text{End}(n_j V_j)^H \simeq \mathfrak{gl}(n_j, \mathbb{H})$  and we have that  $\mathcal{H}(\mathfrak{gl}(n_j, \mathbb{H})) \simeq Z(\mathfrak{gl}(n_j, \mathbb{H})) \simeq \mathbb{R}$ . The theorem follows.  $\square$

#### 4. COMPUTATIONS ON THE QUOTIENT

We now consider the abelianization of the strata preserving vector fields on the quotient  $X/K$ . We recall a few facts about  $X/K$  from [Schw1]. Let  $\pi: X \rightarrow X/K$  denote the canonical map, where  $X/K$  is given the quotient topology. Then  $X/K$  has a differentiable structure where for  $U$  an open subset of  $X/K$ ,  $\mathcal{C}^\infty(U) = \mathcal{C}^\infty(\pi^{-1}(U))^K$ . Let  $H$  be a closed subgroup of  $K$ . Then we have the corresponding stratum  $X^{(H)} := \{x \in X \mid K_x \text{ is conjugate to } H\}$  and its image  $(X/K)^{(H)} \subset X/K$ .

The *isotropy strata*  $(X/K)^{(H)} \subset X/K$  and  $X^{(H)} \subset X$  are smooth and locally closed submanifolds and  $\pi: X^{(H)} \rightarrow (X/K)^{(H)}$  is naturally a smooth fiber bundle (with structure group  $N_K(H)/H$ ). The number of isotropy strata is locally finite on  $X$  and  $X/K$ . Let  $\text{Der}(\mathcal{C}^\infty(X/K))$  denote the derivations of  $\mathcal{C}^\infty(X/K)$  and let  $\mathcal{X}^\infty(X/K)$  denote those derivations that preserve the ideals of functions  $I_{H_i}$  vanishing on the isotropy strata  $(X/K)^{(H_i)}$  of  $X/K$ . Each element of  $\mathcal{X}^\infty(X)^K$  restricts to a derivation of  $\mathcal{C}^\infty(X/K)$ , so there is a canonical map  $\pi_*: \mathcal{X}^\infty(X)^K \rightarrow \text{Der}(\mathcal{C}^\infty(X/K))$ . The main theorem of [Schw1] is that  $\text{Im } \pi_* \subset \mathcal{X}^\infty(X/K)$  and that  $\pi_*$  is surjective. Clearly  $\pi_*$  is a homomorphism of Lie algebras so we have an induced surjection  $\mathcal{H}(\mathcal{X}^\infty(X)^K) \rightarrow \mathcal{H}(\mathcal{X}^\infty(X/K))$ . We only need to compute what happens in the case of  $X = K *^H V$  where  $H$  is a closed subgroup of  $K$  and  $V$  is an  $H$ -module such that  $V^H = (0)$ . Let  $V = \bigoplus_{j=1}^m n_j V_j$  as in Theorem 1.5. The following has Theorem 1.6 as a corollary.

**Theorem 4.1.** *Assume that  $\text{End}(V_j)^H \simeq \mathbb{C}$  if and only if  $j \leq l$  where  $l \leq m$ . Let  $T$  be the corresponding torus  $(S^1)^l \subset \prod_{j=1}^l Z(\text{End}(V_j)^H)$ . Then  $T$  acts on  $V$  commuting with the action of  $H$ , and we have an induced map  $T \rightarrow \text{Aut}(V/H)$ . Let  $S$  denote the kernel where  $\dim S = k$ . Then*

$$\mathcal{H}(\mathcal{X}_c^\infty((K *^H V)/K)) \simeq \mathcal{H}(\mathcal{X}(V/H)) \simeq \mathbb{R}^{m-l+k} \oplus \mathbb{C}^{l-k}.$$

*Proof.* We have the canonical surjection of Lie algebras  $\pi_*: \text{End}(V)^H \rightarrow \mathcal{X}_0(V/H)$  and  $\pi_*$  induces a surjection of  $\mathcal{H}(\text{End}(V)^H)$  onto  $\mathcal{H}(\mathcal{X}(V/H))$ . For every  $j$  we have the identity  $\text{Id}_j \in \text{End}(n_j V_j)^H$  and clearly these elements give linearly independent derivations of  $\mathcal{O}(V)^H$ . Now consider the action of  $T$  on  $V/H$  and its kernel  $S$ . Then  $\mathfrak{s}$  is the kernel of the restriction of  $\pi_*$  to the center of  $\text{End}(V)^H$ , so that  $\mathfrak{s}$  is the kernel on homology.  $\square$

*Example 4.2.* Suppose that  $H$  is a torus acting faithfully on  $V$  and  $V = \sum_{j=1}^m n_j V_j$  where  $V^H = (0)$  as in Theorem 1.5. Then  $\mathfrak{s} \simeq \mathfrak{h}$  and  $\mathcal{H}(\mathcal{X}(V/H)) \simeq \mathbb{R}^k \oplus \mathbb{C}^{m-k}$  where  $k = \dim H$ .

*Example 4.3.* Let  $V = \mathbb{C}^n \oplus \wedge^2 \mathbb{C}^n$  with the canonical action of  $\text{SU}(n, \mathbb{C})$ ,  $n \geq 3$ . Then  $T$  has dimension 2 and  $S$  has dimension 1. See [Schw1, Table I].

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